

# Plane partitions and their pedestal polynomials\*

Oleg Ogievetsky<sup>††</sup> and Senya Shlosman<sup>‡,‡‡</sup>

<sup>‡</sup>Aix Marseille Université, Université de Toulon,  
CNRS, CPT UMR 7332, 13288, Marseille, France

<sup>‡‡</sup>Inst. of the Information Transmission Problems,  
RAS, Moscow, Russia

## Abstract

We define, for an arbitrary partially ordered set, a multi-variable polynomial generalizing the hook polynomial.

## 1 Introduction

Let  $\mathcal{S}$  be a partially ordered set. In this work we associate to  $\mathcal{S}$  a multi-variable polynomial  $\mathfrak{h}$ . When  $\mathcal{S}$  is a Young diagram, the principal specialization of  $\mathfrak{h}$  coincides with the hook polynomial.

Our construction of  $\mathfrak{h}$  begins with defining a polynomial  $\mathfrak{h}_P$ , where  $P$  is an arbitrary linear extension of  $\mathcal{S}$ . Then we show that in fact  $\mathfrak{h}_P$  does not depend on  $P$ . The proof uses the equality (3) (precise definitions are given in sections 2 and 3), which is implied by the bijection between the set of reverse

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<sup>†</sup>On leave of absence from P. N. Lebedev Physical Institute, Leninsky Pr. 53, 117924 Moscow, Russia

partitions on  $\mathcal{S}$  and the product of the set of  $P$ -pedestals on  $\mathcal{S}$  and the set of Young diagrams with at most  $|\mathcal{S}|$  rows. It would be interesting to find a direct, not referring to the formula (3), proof of the theorem 2.

To facilitate the exposition we take for  $\mathcal{S}$  the set of nodes of a Young diagram  $\lambda$ . In this situation, linear extensions of  $\mathcal{S}$  correspond to standard Young tableaux of shape  $\lambda$ , see Definition 1. Our results and proofs work in the same way for general  $\mathcal{S}$  (and linear extensions of  $\mathcal{S}$  instead of standard Young tableaux).

## 2 Main result

Let  $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ ,  $\lambda_1 \geq \dots \geq \lambda_l > 0$ , be a partition of  $n$ ,  $\lambda_1 + \dots + \lambda_l = n$ . We identify  $\lambda$  with its Young diagram, that is, the set of nodes

$$\alpha = (i, j) \text{ with } j = 1, \dots, \lambda_i \text{ for each } i = 1, \dots, l.$$

A *standard Young tableau of shape  $\lambda$*  is a bijection  $Q: \lambda \rightarrow \{1, \dots, n\}$  such that the function  $Q(i, j)$  increases in  $i$  and  $j$ . We denote the set of these standard Young tableaux by  $\mathbf{st}_\lambda$ .

**Definition 1.** Let  $\preceq$  be the minimal transitive partial order on  $\lambda$ , containing  $(i, j) \prec (i+1, j)$  and  $(i, j) \prec (i, j+1)$  for all possible  $(i, j)$ . A standard Young tableau of shape  $\lambda$  can be identified with a linear extension of  $\preceq$ , that is, a linear order, compatible with the partial order  $\preceq$ . We denote by  $\preceq_P$  the linear order associated to a standard tableau  $P$ .

Let  $\mathbb{Z}_{\geq 0}$  be the set of non-negative integers. A *reverse plane partition of shape  $\lambda$*  is a function  $\mathfrak{Q}: \lambda \rightarrow \mathbb{Z}_{\geq 0}$ , non-decreasing in  $i$  and  $j$ . It is *column-strict* if it increases in  $j$ . We visualize reverse plane partitions by placing the number  $\mathfrak{Q}(i, j)$  in the node  $(i, j)$  for each  $(i, j) \in \lambda$ . We denote by  $|\mathfrak{Q}|$  the volume of  $\mathfrak{Q}$ ,  $|\mathfrak{Q}| = \sum_{(i,j) \in \lambda} \mathfrak{Q}(i, j)$ .

Let  $\lambda$  be a partition of  $n$ . Let  $\bar{\mathcal{S}}_\lambda$  be the set of reverse plane partitions of shape  $\lambda$ , and  $\mathcal{S}_\lambda \subset \bar{\mathcal{S}}_\lambda$  the subset of reverse column-strict plane partitions of shape  $\lambda$ . Recall that the Schur function  $s_\lambda$  is the formal power series in infinitely many variables  $\mathbf{x} = (x_0, x_1, x_2, \dots)$ , given by

$$s_\lambda(\mathbf{x}) = \sum_{\mathfrak{Q} \in \mathcal{S}_\lambda} \prod_{\alpha \in \lambda} x_{\mathfrak{Q}(\alpha)}.$$

We need the similarly defined formal power series  $\bar{s}_\lambda$ ,

$$\bar{s}_\lambda(\mathbf{x}) = \sum_{\Omega \in \bar{\mathcal{S}}_\lambda} \prod_{\alpha \in \lambda} x_{\Omega(\alpha)}.$$

The Schur function  $s_\lambda(\mathbf{x})$ , unlike our ‘wrong’ Schur function,  $\bar{s}_\lambda(\mathbf{x})$ , is symmetric (see, e.g. [3]).

To define the pedestal polynomial we proceed as follows. Let  $P, Q \in \mathfrak{st}_\lambda$  be two standard Young tableaux of shape  $\lambda$ . We are going to compare the corresponding linear orders  $\preceq_P$  and  $\preceq_Q$  on the set of the nodes of  $\lambda$ . Let  $\alpha_1 = (1, 1) \prec_Q \alpha_2 \prec_Q \cdots \prec_Q \alpha_n$  be the list of all the nodes of  $\lambda$ , enumerated according to the order  $\preceq_Q$ . We say that a node  $\alpha_k$  is a  $(P, Q)$ -disagreement node, if  $\alpha_{k+1} \prec_P \alpha_k$  (while, of course,  $\alpha_k \prec_Q \alpha_{k+1}$ ). We define the reverse plane partition  $q_{P,Q}$  of shape  $\lambda$  by

$$q_{P,Q}(\alpha_k) = \# \{l : l < k, \alpha_l \text{ is a } (P, Q)\text{-disagreement node}\}. \quad (1)$$

Indeed, the function  $q_{P,Q}$  is non-decreasing with respect to the order  $\preceq_Q$ , hence  $q_{P,Q}$  is a reverse plane partition.

The reverse plane partition  $q_{P,Q}$  thus defined is called  $P$ -pedestal of  $Q$ , see [2]. We finally define the polynomial  $\mathfrak{h}_P(\mathbf{x})$  by

$$\mathfrak{h}_P(\mathbf{x}) = \sum_{Q \in \mathfrak{st}_\lambda} \prod_{\alpha \in \lambda} x_{q_{P,Q}(\alpha)}. \quad (2)$$

The set (as  $Q$  runs over  $\mathfrak{st}_\lambda$ ) of  $P$ -pedestals depends on  $P$ . However, the polynomial  $\mathfrak{h}_P$  has the following remarkable property.

**Theorem 2.** *The function  $\mathfrak{h}_P(\mathbf{x})$  does not depend on  $P$  from  $\mathfrak{st}_\lambda$ .*

**Definition 3.** *We call the function*

$$\mathfrak{h}_\lambda(\mathbf{x}) = \mathfrak{h}_P(\mathbf{x}),$$

*where  $P$  is any standard Young tableau  $P$  of shape  $\lambda$ , the **pedestal polynomial**.*

Our theorem states that the pedestal polynomial is well-defined. For example,  $\mathfrak{h}_{(3,2)}(\mathbf{x}) = x_0^5 + x_0^4 x_1 + x_0^3 x_1^2 + x_0^2 x_1^3 + x_0^2 x_1^2 x_2$ .

### 3 Proof and discussion

*Proof of the Theorem.* Let  $R_n$  be the component of degree  $n$  of the ring of formal power series in variables  $\mathbf{x} = (x_0, x_1, x_2, \dots)$  with integer coefficients. The element  $u(\mathbf{x})$  of  $R_n$  is a sum,

$$u(\mathbf{x}) = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} a_{i_1 i_2 \dots i_n} \mathbf{x}_{i_1, i_2, \dots, i_n}, \quad \mathbf{x}_{i_1, i_2, \dots, i_n} := x_{i_1} x_{i_2} \dots x_{i_n},$$

where  $i_1, i_2, \dots, i_n$  are non-negative integers, and the coefficients  $a_{i_1 i_2 \dots i_n}$  are integer. For example, the functions  $s_\lambda(\mathbf{x})$  and  $\bar{s}_\lambda(\mathbf{x})$  belong to  $R_n$ . The function  $\bar{s}_{(n)}$ , corresponding to the one-row partition  $\lambda = (n)$ ,

$$\bar{s}_{(n)}(\mathbf{x}) = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} \mathbf{x}_{i_1, i_2, \dots, i_n} \in R_n$$

will play a special role in our argument.

We define the  $*$ -product on monomials by

$$\mathbf{x}_{i_1, i_2, \dots, i_n} * \mathbf{x}_{j_1, j_2, \dots, j_n} := \mathbf{x}_{i_1 + j_1, i_2 + j_2, \dots, i_n + j_n}$$

and extend it by linearity to the ring structure on  $R_n$ . The ring  $(R_n, *)$  is isomorphic to a subring of the ring  $\mathbb{C}[[y_1, \dots, y_n]]$  of formal power series in  $n$  variables, via the monomorphism  $\varphi_n : R_n \rightarrow \mathbb{C}[[y_1, \dots, y_n]]$ , defined by

$$\varphi_n(\mathbf{x}_{i_1, i_2, \dots, i_n}) = y_1^{i_1} \dots y_n^{i_n}.$$

In particular,  $(R_n, *)$  inherits from  $\mathbb{C}[[y_1, \dots, y_n]]$  the property of having no zero divisors.

Fix a standard Young tableau  $P \in \mathfrak{st}_\lambda$ . We will prove now the identity

$$\bar{s}_\lambda(\mathbf{x}) = \mathfrak{h}_P(\mathbf{x}) * \bar{s}_{(n)}(\mathbf{x}), \tag{3}$$

which implies, due to the absence of zero divisors in  $(R_n, *)$ , the assertion of the theorem, since the first and the last terms in the identity do not depend on  $P$ .

The bijective proof of the identity (3) follows from [2]. Relations (46), (48) and (50) of that paper describe bijections  $b_{st}$ ,  $b_{st}^{-1}$  between the set of reverse plane partitions of shape  $\lambda$  and the product of the set of  $P$ -pedestals and the set of Young diagrams with at most  $n$  rows. Let  $\mathfrak{Q}$  be a reverse

plane partition and  $b_{St}(\mathfrak{Q}) = (q, \mu)$ , where  $q$  is a  $P$ -pedestal and  $\mu$  a Young diagram. The construction of  $b_{St}$  (see below) implies that the monomial, corresponding to  $\mathfrak{Q}$  in  $\bar{s}_\lambda(\mathbf{x})$  is the  $*$ -product of the monomial corresponding to  $q$  in  $\mathfrak{h}_P(\mathbf{x})$  and the monomial corresponding to  $\mu$  in  $\bar{s}_{(n)}(\mathbf{x})$ , and the proof of (3) follows. These bijections were in fact used earlier by D. Knuth in [1].

The bijections  $b_{St}, b_{St}^{-1}$  are defined as follows.

Let  $\mathfrak{Q}$  be a reverse plane partition of shape  $\lambda$ . Then we can define the partition of  $|\mathfrak{Q}|$  with at most  $n$  rows,

$$p = \Pi(\mathfrak{Q}), \quad (4)$$

by just listing all the entries of the two-dimensional array of values of  $\mathfrak{Q}$  in the non-increasing order.

To every reverse plane partition  $\mathfrak{Q}$  we associate the standard Young tableau  $Q(\mathfrak{Q}) \in \mathfrak{st}_\lambda$  as follows. Define the linear order  $\prec_{\mathfrak{Q}}$  on the nodes of  $\lambda$  by

$$\alpha' \prec_{\mathfrak{Q}} \alpha'' \text{ if } \mathfrak{Q}(\alpha') < \mathfrak{Q}(\alpha'') \text{ or if } \mathfrak{Q}(\alpha') = \mathfrak{Q}(\alpha'') \text{ and } \alpha' \prec_P \alpha''.$$

Then  $Q(\mathfrak{Q})$  is defined by the relation:  $\prec_{Q(\mathfrak{Q})} = \prec_{\mathfrak{Q}}$ . Now the bijection  $b_{St}$  is defined by

$$b_{St}(\mathfrak{Q}) = \left( q_{P, Q(\mathfrak{Q})}, \Pi \left( \mathfrak{Q} - q_{P, Q(\mathfrak{Q})} \right) \right),$$

where the reverse plane partition  $\mathfrak{Q} - q_{P, Q(\mathfrak{Q})}$  is given by

$$\left( \mathfrak{Q} - q_{P, Q(\mathfrak{Q})} \right)(\alpha) = \mathfrak{Q}(\alpha) - q_{P, Q(\mathfrak{Q})}(\alpha), \quad \alpha \in \lambda.$$

To construct the inverse bijection,  $b_{St}^{-1}$ , we first associate to every standard Young tableau  $Q \in \mathfrak{st}_\lambda$  and every partition  $p = (\Lambda_1, \Lambda_2, \dots, \Lambda_n)$  the reverse plane partition  $\mathfrak{Q}_{Q,p}$  of shape  $\lambda$ , by

$$\mathfrak{Q}_{Q,p}(i, j) = \Lambda_{Q(i,j)}.$$

Then

$$b_{St}^{-1}(q_{P,Q}, p) = q_{P,Q} + \mathfrak{Q}_{Q,p}.$$

The proof is finished.  $\square$

The identity (3) is of independent interest. The principal specialization,  $x_i \mapsto x^i$ , turns the ‘wrong’ Schur function  $\bar{s}_\lambda(\mathbf{x})$  into the generating function

$\sigma_\lambda(x)$  for the number of reverse plane partitions of shape  $\lambda$ , given by the Mac-Mahon–Stanley formula

$$\sigma_\lambda(x) = \frac{1}{\prod_{\alpha \in \lambda} (1 - x^{h_\alpha})},$$

where  $h_\alpha$  is the hook length of a node  $\alpha$  of  $\lambda$ . The term  $\bar{s}_{(n)}(\mathbf{x})$  turns into the generating function of Young diagrams with at most  $n$  rows:

$$\sigma_{(n)}(x) = \frac{1}{(1-x)(1-x^2)\dots(1-x^n)}.$$

Finally, the function  $\mathfrak{h}_\lambda(\mathbf{x})$  turns into the generating polynomial  $\pi_\lambda(x)$  of the sequence  $\{p_{\lambda,k}\}$  with  $p_{\lambda,k}$  the number of  $P$ -pedestals of volume  $k \geq 0$  for some  $P \in \mathfrak{st}_\lambda$ . We obtain

$$\frac{1}{\prod_{\alpha \in \lambda} (1 - x^{h_\alpha})} = \frac{\pi_\lambda(x)}{(1-x)(1-x^2)\dots(1-x^n)}. \quad (5)$$

It follows from (2) that the function

$$\pi_\lambda(x) = \sum_{Q \in \mathfrak{st}_\lambda} x^{|q_{P,Q}|} \quad (6)$$

does not depend on the choice of  $P$  while the contribution of an individual standard Young tableau  $Q$  does.

The formula (5) can be found in [3], but there the polynomial  $\pi_\lambda(x)$  is given by any of two other expressions:

$$\pi_\lambda(x) = x^{-l(\lambda)} \sum_{Q \in \mathfrak{st}_\lambda} x^{\text{maj}(Q)} \text{ and } \pi_\lambda(x) = x^{-l(\lambda)} \sum_{Q \in \mathfrak{st}_\lambda} x^{\text{comaj}(Q)},$$

where  $l(\lambda) = \sum_{(i,j) \in \lambda} (i-1)$ . It is interesting to note that in general neither of the two functions on  $\mathfrak{st}_\lambda$ ,  $\text{maj}(\cdot) - l(\lambda)$  and  $\text{comaj}(\cdot) - l(\lambda)$ , nor their partners for the transposed to  $\lambda$  Young diagram, belong to our family  $\{\text{vol}(q_{P,*}) : P \in \mathfrak{st}_\lambda\}$ . For example, take  $\lambda = (3, 2, 1)$ .

## References

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